

ON PURELY INSEPARABLE ALGEBRAS AND P.H.D. RINGS

BY

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ABSTRACT. M. E. Sweedler has considered purely inseparable algebras over rings. We define a stronger notion for purely inseparable algebras over rings and we study the fundamental properties of purely inseparable algebras. Moreover, we consider the relations between purely inseparable algebras and P.H.D. rings.

Introduction. Let K be a commutative ring with 1 and let R be a K -algebra. M. E. Sweedler has said that R is a purely inseparable K -algebra if the multiplication map $\gamma_R: R \otimes_K R \rightarrow R$ gives an $R \otimes_K R$ -module projective cover of R [8]. But we shall define in this paper a purely inseparable K -algebra as follows. R is a purely inseparable K -algebra if there exists at most one K -algebra homomorphism of R into any reduced K -algebra. Our notion is stronger than the notion of M. E. Sweedler. But if K is a field, then both notions are identical.

In §1 we shall show R is a purely inseparable K -algebra if and only if $\ker \gamma_R$ is a nil ideal. This implies the fundamental properties of purely inseparable K -algebras. Let K be a commutative ring of characteristic $p > 0$ and let \mathfrak{S} be the set of prime ideals \mathfrak{p} of K such that $\mathfrak{p}R \neq R$. Suppose R is a flat K -module. Then R is a purely inseparable K -algebra if and only if, for every $\mathfrak{p} \in \mathfrak{S}$, there exists a multiplicatively closed subset \bar{S} of $K \cdot 1_R$ satisfying $\bar{S} \cap \mathfrak{p}1_R = \emptyset$ and, for every $x \in R$, we have $ax^{p^n} \in K \cdot 1_R$ for some integer n and for some element $a \in K$ such that $a \cdot 1_R \in \bar{S}$. Next we give a structure theorem for purely inseparable algebras of finite exponent. Let K be a commutative ring of characteristic $p > 0$ and let R be a purely inseparable K -algebra of finite exponent with the structure morphism ϕ . Let $\text{Spec}(K/R) = \{\mathfrak{p} \in \text{Spec}(K); \phi^{-1}(\mathfrak{p}R) = \mathfrak{p}\}$ and let ψ be a mapping of $\text{Spec}(R)$ into $\text{Spec}(K)$ such that $\psi(\mathfrak{P}) = \phi^{-1}(\mathfrak{P})$ for $\mathfrak{P} \in \text{Spec}(R)$. Then ψ is a bijection of $\text{Spec}(R)$ onto $\text{Spec}(K/R)$. Moreover, if R is K -projective, then we have (1) $\text{Spec}(K) = \text{Spec}(K/R)$ and (2) R is a faithfully flat K -module.

In §2 we introduce a notion of P.H.D. rings. We shall say that a K -algebra R is a P.H.D. ring over K with respect to an R -module M if $\text{Hom}_K(R, M)$ is filled with all the high order derivations of R into M . A K -algebra R is called a P.H.D. ring over K if R is a P.H.D. ring over K with respect to all R -modules. The case in which K is a field was investigated in [6]. In this paper we shall delete the assumption K is a field. The following are equivalent. (1) R is a P.H.D. ring over K . (2) R is a P.H.D.

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ring over K with respect to $R \otimes_K R$. (3) $(\ker \gamma_R)^n = 0$ for some integer n . (4) For any K -algebra T and any K -algebra homomorphisms u_1, u_2 of R into T , there exists an integer n such that

$$\prod_{i=1}^n (u_1(x_i) - u_2(x_i)) = 0 \quad \text{for any } n\text{-elements } x_i \in R.$$

This implies the fundamental properties of P.H.D. rings over K . Suppose R is a projective K -module. If R is a P.H.D. ring over K with respect to R , then R is a purely inseparable K -algebra. Moreover if R is a free K -module, then a P.H.D. ring over K with respect to R is a P.H.D. ring over K . We shall give a structure theorem. Suppose R is a finite flat K -module and suppose R is a P.H.D. ring over K with respect to R with the structure homomorphism ϕ . Let ψ be a mapping of $\text{Spec}(R)$ into $\text{Spec}(K)$ such that $\psi(\mathfrak{P}) = \phi^{-1}(\mathfrak{P})$ for $\mathfrak{P} \in \text{Spec}(R)$. Then it holds that (1) ψ is bijective, and (2) R is a faithfully flat K -module.

In what follows all rings and algebras are *commutative with identities*, and all modules and all homomorphisms are unitary.

1. Purely inseparable K -algebras.

DEFINITION 1. Let K be a commutative ring and let R be a K -algebra. It is said that R is a purely inseparable K -algebra if there exists at most one K -algebra homomorphism of R into any reduced K -algebra.

Say $\phi_i: R \rightarrow S$ are K -algebra homomorphisms for $i = 1, 2$ where R is a purely inseparable K -algebra and S is a K -algebra. Let N be the ideal of all nilpotent elements of S and $\pi: S \rightarrow S/N$. Then $\pi \cdot \phi_1 = \pi \cdot \phi_2$ since S/N is reduced. Hence, for all $r \in R$: $\phi_1(r) - \phi_2(r)$ is nilpotent.

Let K be a commutative ring and let R be a K -algebra. Let $I_{R/K}$ denote the kernel of the multiplication map $\gamma_R: R \otimes_K R \rightarrow R$.

THEOREM 1. Let K be a commutative ring and let R be a K -algebra. R is a purely inseparable K -algebra if and only if $I_{R/K}$ is a nil ideal.

PROOF. Assume R is a purely inseparable K -algebra. Let u_1, u_2 be K -algebra homomorphisms of R into $R \otimes_K R$ such that $u_1(x) = x \otimes 1_R$, $u_2(x) = 1_R \otimes x$. Then for all $x \in R$: $u_1(x) - u_2(x)$ is nilpotent. Therefore, $(x \otimes 1_R - 1_R \otimes x)$ is nilpotent and thus $I_{R/K}$ is a nil ideal. Conversely, assume $I_{R/K}$ is a nil ideal and R is not a purely inseparable K -algebra. Then there exist a reduced K -algebra T and distinct K -algebra homomorphisms u_1, u_2 of R into T . Then there is an element $x_0 \in R$ such that $u_1(x_0) \neq u_2(x_0)$. Consider a K -linear mapping $u_1 \otimes u_2$ of $R \otimes_K R$ into T defined by $(u_1 \otimes u_2)(x \otimes y) = u_1(x)u_2(y)$. Then $u_1 \otimes u_2$ is a K -algebra homomorphism. Since $I_{R/K}$ is a nil ideal, there is an integer m such that

$$(x_0 \otimes 1_R - 1_R \otimes x_0)^m = 0.$$

Thus we have

$$0 = (u_1 \otimes u_2)[(x_0 \otimes 1_R - 1_R \otimes x_0)^m] = (u_1(x_0) - u_2(x_0))^m.$$

Since T is reduced, this is a contradiction. Therefore, R is a purely inseparable K -algebra. Q.E.D.

Our notion is stronger than that of M. E. Sweedler. But when K is a field, both notions are identical by Theorem 11 in [8].

PROPOSITION 1. *Let K be a commutative ring and let R, S be K -algebras. Then we have the following:*

(a) *If R is a purely inseparable K -algebra and $S \rightarrow R$ is a K -algebra homomorphism, then R is a purely inseparable S -algebra.*

(b) *If R is a purely inseparable K -algebra, then $R \otimes_K S$ is a purely inseparable S -algebra.*

(c) *If R and S are purely inseparable K -algebras, then $R \otimes_K S$ is a purely inseparable K -algebra.*

PROOF. (a) is easily seen from Definition 1.

(b) Let T be a reduced S -algebra and let u_1, u_2 be S -algebra homomorphisms of $R \otimes_K S$ into T . Since R is a purely inseparable K -algebra, we have $u_1(x \otimes 1_S) = u_2(x \otimes 1_S)$ for all $x \in R$. For any $y \in S$ we have

$$u_1(x \otimes y) - u_2(x \otimes y) = (u_1(x \otimes 1_S) - u_2(x \otimes 1_S))y = 0$$

and therefore, $R \otimes_K S$ is a purely inseparable S -algebra.

(c) Let T be a reduced K -algebra and let u_1, u_2 be K -algebra homomorphisms of $R \otimes_K S$ into T . Since R, S are purely inseparable K -algebras, it follows that $u_1(x \otimes 1_S) = u_2(x \otimes 1_S)$ [resp. $u_1(1_R \otimes y) = u_2(1_R \otimes y)$] for any $x \in R$ [resp. $y \in S$]. Thus we have $u_1 = u_2$ and hence $R \otimes_K S$ is a purely inseparable K -algebra. Q.E.D.

PROPOSITION 2. *Let K be a commutative ring and let R, R' be K -algebras where $\pi: R \rightarrow R'$ is a surjective map of K -algebras. If R is a purely inseparable K -algebra then R' is a purely inseparable K -algebra. If $\ker \pi$ is a nil ideal, then the converse is true.*

PROOF. Let T be a reduced K -algebra. Consider the commutative diagram:

$$\begin{array}{ccccc} R & \xrightarrow{\pi} & R' & \xrightarrow[u_2]{u_1} & T \\ & \nearrow & & & \\ K & & & & \end{array}$$

Let u_1, u_2 be K -algebra homomorphisms of R' into T . Since R is a purely inseparable K -algebra, we have $u_1\pi(x) = u_2\pi(x)$ for all $x \in R$. Since π is surjective, R' is a purely inseparable K -algebra. Let us prove the converse under the assumption that $\ker \pi$ is nil. Consider the commutative exact diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & I_{R/K} & \rightarrow & R \otimes_K R & \rightarrow & R \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & I_{R'/K} & \rightarrow & R' \otimes_K R' & \rightarrow & R' \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Let $\sum x_i \otimes y_i \in I_{R/K}$. Then by assumption we have $[(\pi \otimes \pi)(\sum x_i \otimes y_i)]^n = 0$ and hence $[\sum (x_i \otimes y_i)]^n \in \ker(\pi \otimes \pi)$. Since $\ker \pi$ is a nil ideal, $\sum x_i \otimes y_i$ is nilpotent and hence R is a purely inseparable K -algebra by Theorem 1. Q.E.D.

COROLLARY. *Let K be a commutative ring and let R be a K -algebra. If R is a purely inseparable K -algebra and \mathfrak{M} is a maximal ideal of K such that $\mathfrak{M}R \neq R$, then $R/\mathfrak{M}R$ is a quasi-local ring.*

PROOF. By Proposition 2, $R/\mathfrak{M}R$ is a purely inseparable K -algebra. Hence $R/\mathfrak{M}R$ is a purely inseparable K/\mathfrak{M} -algebra by Proposition 1 (a). Since K/\mathfrak{M} is a field, $R/\mathfrak{M}R$ is a quasi-local ring by Theorem 11 in [8]. Q.E.D.

PROPOSITION 3. *Let K be a commutative ring and let R be a K -algebra. Then the following are equivalent.*

- (1) R is a purely inseparable K -algebra.
- (2) For any maximal ideal \mathfrak{M} of K , $R_S = R \otimes_K K_S$ is a purely inseparable K_S -algebra ($S = K - \mathfrak{M}$).
- (3) For any prime ideal \mathfrak{p} of K , $R_S = R \otimes_K K_S$ is a purely inseparable K_S -algebra ($S = K - \mathfrak{p}$).

PROOF. Since K_S is K -flat, we have $I_{R_S/K_S} = S^{-1}I_{R/K}$. Thus we have (1) \rightarrow (2), (1) \rightarrow (3) by Theorem 1. Evidently (3) \rightarrow (2). Thus it is sufficient to show (2) \rightarrow (1). For every $x \in R$ let us set $\mathfrak{A}_x = \{s \in K: s(1_R \otimes x - x \otimes 1_R)^n = 0 \text{ for some integer } n\}$. Then \mathfrak{A}_x is an ideal of K . Suppose $\mathfrak{A}_x \neq K$. Then there is a maximal ideal \mathfrak{M} of K containing \mathfrak{A}_x . By assumption R_S is a purely inseparable K_S -algebra ($S = K - \mathfrak{M}$) and hence by Theorem 1, there is an integer n such that

$$(1_R \otimes x - x \otimes 1_R)^n = 0$$

in $R_S \otimes_{K_S} R_S$. It follows $s(1_R \otimes x - x \otimes 1_R)^n = 0$ in $R \otimes_K R$ for some $s \in S$. This shows $s \in \mathfrak{A}_x$ and this is a contradiction. Therefore, $\mathfrak{A}_x = K$ and R is a purely inseparable K -algebra by Theorem 1. Q.E.D.

PROPOSITION 4 (TRANSITIVITY). *Let K be a commutative ring. Let L, R be K -algebras and let $u: L \rightarrow R$ be a K -algebra homomorphism making R a purely inseparable L -algebra. If L is a purely inseparable K -algebra, then R is a purely inseparable K -algebra.*

PROOF. For each element $x \in I_{R/K}$ there is an integer n satisfying $v(x)^n = 0$ where v is the canonical mapping of $R \otimes_K R$ into $R \otimes_L R$. Since v is a ring homomorphism, we have $v(x^n) = 0$ and hence $x^n \in \ker v = I_{L/K}(R \otimes_K R)$. Therefore, x is nilpotent and hence R is a purely inseparable K -algebra by Theorem 1. Q.E.D.

COROLLARY. *Let K be a commutative ring and let R be a K -algebra. Let $Q(K), Q(R)$ be the total quotient rings of K, R respectively. If R is a purely inseparable K -algebra, then $Q(R)$ is a purely inseparable $Q(K)$ -algebra.*

PROOF. $Q(R)$ is a purely inseparable R -algebra. Thus by Proposition 4, $Q(R)$ is a purely inseparable K -algebra and $Q(R)$ is a purely inseparable $Q(K)$ -algebra by Proposition 1(a). Q.E.D.

DEFINITION 2. Let K be a commutative ring and let R be a K -algebra. We say R is K -epimorphic if for every K -algebra T and for every K -algebra homomorphisms u_1, u_2 of R into T the relation $u_1\phi = u_2\phi$ always implies $u_1 = u_2$ where ϕ is the structure homomorphism of R (cf. [7]).

By Proposition 3 in [7], R is K -epimorphic if and only if we have $1 \otimes x = x \otimes 1$ in $R \otimes_K R$ for all $x \in R$.

THEOREM 2. Let K be a commutative ring and let R be a K -algebra. R is K -epimorphic if and only if R is a separable and purely inseparable K -algebra.

PROOF. The if part is obvious from Corollary 7(a) in [8]. Conversely, assume R is K -epimorphic. It is sufficient to show R is a separable K -algebra. By the rule $(a \otimes b)x = abx$ for $a, b, x \in R$, we can make R an $R \otimes_K R$ -module. Then γ_R is an $R \otimes_K R$ -homomorphism of $R \otimes_K R$ onto R . Since R is K -epimorphic, γ_R is bijective. Hence γ_R is an $R \otimes_K R$ -isomorphism and therefore, R is a free $R \otimes_K R$ -module. Thus R is a separable K -algebra. Q.E.D.

COROLLARY (TRANSITIVITY). Let K be a commutative ring. Let L be a K -algebra and let R be an L -algebra. If L is K -epimorphic and R is L -epimorphic, then R is K -epimorphic.

THEOREM 3. Let K be a commutative ring of characteristic $p > 0$ and let R be a K -algebra with the structure homomorphism ϕ . Let \mathfrak{S} be the set of prime ideals \mathfrak{p} of K such that $\mathfrak{p}R \neq R$. Suppose R is K -flat. Then R is a purely inseparable K -algebra if and only if, for every $\mathfrak{p} \in \mathfrak{S}$, there exists a multiplicatively closed subset \bar{S} of $\phi(K)$ such that (1) $\bar{S} \cap \phi(\mathfrak{p}) = \emptyset$ and (2) for every $x \in R$, we have $ax^{p^n} \in \phi(K)$ for some integer n and for some element $a \in K$ satisfying $\phi(a) \in \bar{S}$. Moreover, R is a purely inseparable K -algebra if there exists an integer n such that $x^{p^n} \in Q(K) \cdot 1_R$ for every element $x \in R$.

PROOF. Assume R is a purely inseparable K -algebra. Let \mathfrak{p} be any element of \mathfrak{S} and let x be any element of R . Then by Theorem 1 we have $(x \otimes 1_R - 1_R \otimes x)^{p^n} = 0$ for some integer n . By the famous lemma of Bourbaki [2, §2, Proposition 13] there exist elements $a_{ji} \in K$ and $x_j \in R$ such that

$$a_{j1} \cdot 1_R + a_{j2}x^{p^n} = 0 \quad \text{for all } j, \quad (1)$$

$$x^{p^n} = \sum x_j a_{j1}, \quad -1_R = \sum x_j a_{j2}. \quad (2)$$

Let $\bar{\mathfrak{p}}$ be a prime ideal of R containing $\phi(\mathfrak{p})$ and let $\bar{T} = R - \bar{\mathfrak{p}}$. Put $\bar{S} = \phi(\phi^{-1}(\bar{T}))$. Then from the second formula in (2), there is an integer j_0 such that $\phi(a_{j_0 2}) \notin \bar{\mathfrak{p}}$. With $a = a_{j_0 2}$ formula (1) shows that a, \bar{S} are as required. Let us prove the converse. Assume R is not a purely inseparable K -algebra. Then there exist a reduced K -algebra T and distinct K -algebra homomorphisms u_1, u_2 of R into T . Let x_0 be an element of R satisfying $u_1(x_0) \neq u_2(x_0)$. Let us set $\mathfrak{A} = \{s \in K; su_1(x_0) = su_2(x_0)\}$. Then \mathfrak{A} is an ideal of K and $\mathfrak{A}R \neq R$. Hence there is a prime ideal $\bar{\mathfrak{p}}$ of R containing $\mathfrak{A}R$. Since $\phi^{-1}(\bar{\mathfrak{p}})R \neq R$, by assumption there exists a multiplicatively closed subset \bar{S} of $\phi(K)$ such that $\bar{S} \cap \phi(\phi^{-1}(\bar{\mathfrak{p}})) = \emptyset$. Moreover, there exist an element $a \in K$ and an integer n such that $\phi(a) \in \bar{S}$, $ax_0^{p^n} \in \phi(K)$. Thus we have

$u_1(ax_0^{p^n}) = u_2(ax_0^{p^n})$. Since T is reduced, it holds that $au_1(x_0) = au_2(x_0)$ and $a \in \mathfrak{A}$. This implies $\phi(a) \in \bar{\mathfrak{p}}$. This is a contradiction. Therefore, R is a purely inseparable K -algebra. We prove the last assertion. Since R is a flat K -module, we have an exact sequence:

$$0 \rightarrow R \otimes_K R \rightarrow Q(K) \otimes_K R \otimes_K R.$$

By assumption we have $x^{p^n} \otimes 1_R - 1_R \otimes x^{p^n} = 0$ in $Q(K) \otimes_K R \otimes_K R$ and hence in $R \otimes_K R$. Therefore, R is a purely inseparable K -algebra by Theorem 1. Q.E.D.

The last assertion of Theorem 3 is not necessarily true if R is not K -flat. We shall give an example.

EXAMPLE 1. Let K be a local domain of characteristic 2 and let $R = K \oplus Kx$ where $x^2 = x$ and $\alpha x = 0$ for any noninvertible element α of K . Then $x = 0$ in $Q(K)R$, but R is not a purely inseparable K -algebra.

REMARK. Let K be a commutative ring of characteristic $p > 0$ and let R be a K -algebra with the structure homomorphism ϕ . Let \mathfrak{p} be a prime ideal of K . If there exists a multiplicatively closed subset \bar{S} of $\phi(K)$ satisfying $\bar{S} \cap \phi(\mathfrak{p}) = \emptyset$ such that for every $x \in R$ we have $tx^{p^n} \in \phi(K)$ for some $\phi(t) \in \bar{S}$ and for some integer n , then we have $\mathfrak{p}R \neq R$. For, suppose $\mathfrak{p}R = R$; then there exist elements $a_i \in \mathfrak{p}$ and $x_i \in R$ such that $1_R = \sum a_i x_i$. By assumption we have $t_i x_i^{p^n} \in \phi(K)$ for some integer n and for some elements $t_i \in K$ satisfying $\phi(t_i) \in \bar{S}$. Thus we have

$$\phi\left(\prod t_i\right) = \sum a_i^{p^n} \left(\prod_{j \neq i} t_j\right) t_i x_i^{p^n} \in \phi(\mathfrak{p})$$

and this is a contradiction. Therefore we have $\mathfrak{p}R \neq R$.

COROLLARY 1. Let K be a commutative ring of characteristic $p > 0$ and let R be a K -algebra. Suppose K is a quasi-local ring with the maximal ideal \mathfrak{M} such that $\mathfrak{M}R \neq R$ and R is K -flat. Then if R is a purely inseparable K -algebra, then any sub- K -algebra L of R is a purely inseparable K -algebra and quasi-local ring.

PROOF. Apply Theorem 3 for the maximal ideal \mathfrak{M} . Then \bar{S} in Theorem 3 is a set of units in $\phi(K)$. Thus for any $x \in R$, we have $x^{p^n} \in \phi(K)$ for some integer n and hence L is a purely inseparable K -algebra by Theorem 1. Moreover, it is trivial that the radical of $\phi(\mathfrak{M})$ in L is the maximal ideal of L . Therefore, L is a quasi-local ring. Q.E.D.

COROLLARY 2. Let K be a commutative ring of characteristic $p > 0$ and let R be a K -algebra with the structure homomorphism ϕ . Suppose R is a flat K -module. Then the following are equivalent.

- (1) R is purely inseparable and faithfully flat.
- (2) For every prime ideal \mathfrak{p} of K , there exists a multiplicatively closed subset \bar{S} of $\phi(K)$ satisfying $\bar{S} \cap \phi(\mathfrak{p}) = \emptyset$ such that for any element $x \in R$, we have $tx^{p^n} \in \phi(K)$ for some integer n and for some element $t \in K$ satisfying $\phi(t) \in \bar{S}$.
- (3) For every element $x \in R$, there exists an integer n such that $x^{p^n} \in \phi(K)$.

PROOF. Since R is faithfully flat, we have $\mathfrak{M}R \neq R$ for all maximal ideals \mathfrak{M} of K . Therefore, the implication (1) \rightarrow (2) follows from Theorem 3. The converse

(2) \rightarrow (1) follows from Theorem 3 and Remark. (3) \rightarrow (2) is trivial. Let us prove (2) \rightarrow (3). Let x be any element of R and let $\mathfrak{A}_x = \{s \in K: sx^{p^n} \in \phi(K) \text{ for some integer } n\}$. Suppose $\mathfrak{A}_x \neq K$. Then \mathfrak{A}_x is an ideal of K and hence there is a prime ideal \mathfrak{p} containing \mathfrak{A}_x . By assumption there exists an element $t \in K - \mathfrak{A}_x$ such that $tx^{p^n} \in \phi(K)$ for some integer n . This is a contradiction. Therefore $\mathfrak{A}_x = K$ and (2) implies (3). Q.E.D.

Let K be a commutative ring of characteristic $p > 0$ and let R be a purely inseparable K -algebra. If there exists an integer n such that $x^{p^n} \otimes 1_R - 1_R \otimes x^{p^n} = 0$ for all elements $x \in R$, then we shall call R a finite exponent n over K . We shall give a structure theorem for purely inseparable K -algebras of finite exponent.

LEMMA 1. *Let K be a commutative ring of characteristic $p > 0$ and let R be a purely inseparable K -algebra of finite exponent n . Let $t(n)$ be the K^{p^n} -torsion submodule of R^{p^n} and let t be the K -torsion submodule of R where $R^{p^n} = \{x^{p^n}; x \in R\}$. If K is a domain, then we have (1) $t(n) = R^{p^n}$ or $t(n)$ is a prime ideal of R^{p^n} , (2) $\sqrt{t} = \sqrt{t(n)}$ where $\sqrt{t(n)}$, \sqrt{t} are radicals of $t(n)$ and t in R respectively, and (3) $\sqrt{t} = R$ or \sqrt{t} is a prime ideal of R .*

PROOF. When there is an element $a \in K$ such that $a \cdot 1_R = 0$, we have $t(n) = R^{p^n}$ and $t = R$. Suppose $a \cdot 1_R \neq 0$ for any element $a \in K$; then $Q(K)$ is isomorphic to $Q(K) \cdot 1_R$, and take these as identifications. (1) Let f be a mapping of R^{p^n} into $R \otimes_K Q(K)$ such that $f(x^{p^n}) = x^{p^n} \otimes 1$. Since the center of the tensor algebra $T_{Q(K)}(R \otimes_K Q(K))$ is $Q(K)$ and $T_{Q(K)}(R \otimes_K Q(K)) \cong T_K(R) \otimes Q(K)$, it holds that $f(x^{p^n}) \in Q(K)$ by assumption. Since $Q(K)$ is a field, $\ker f$ is a prime ideal. We have

$$\begin{aligned} x^{p^n} \in \ker f &\Leftrightarrow \text{there exists a nonzero element} \\ &\quad r \in K \text{ such that } r^{p^n} x^{p^n} = 0 \text{ in } R \\ &\Leftrightarrow x^{p^n} \in t(n). \end{aligned}$$

Hence $t(n) = \ker f$ is a prime ideal of R^{p^n} . Let us prove $\sqrt{t(n)}$ is a prime ideal of R . Let $xy \in \sqrt{t(n)}$, $x \notin \sqrt{t(n)}$ for $x, y \in R$. Then we have $(x^{p^n})^m (y^{p^n})^m \in t(n)$ for some integer m . Since $t(n)$ is a prime ideal of R^{p^n} , we have $(y^{p^n})^m \in t(n)$ and hence $\sqrt{t(n)}$ is a prime ideal of R . To show $\sqrt{t} = \sqrt{t(n)}$ it is sufficient to prove $t \subseteq \sqrt{t(n)}$ since $\sqrt{t(n)}$ is a prime ideal of R . Let $x \in t$. Then we have $s'x = 0$ for some element $s' \in K$, $s' \neq 0$. Since K is a domain, we have $s'^{p^n} \neq 0$ and $s'^{p^n} x^{p^n} = 0$. Thus $x \in \sqrt{t(n)}$ and this shows $\sqrt{t} = \sqrt{t(n)}$. Q.E.D.

LEMMA 2. *Under the same assumptions and notations as in Lemma 1, \sqrt{t} is the unique maximal element in the set of ideals \mathfrak{A} of R such that $Q(K)\mathfrak{A} \cap Q(K) \cdot 1_R = 0$.*

PROOF. When there is an element $a \in K$ such that $a \cdot 1_R = 0$, we have $Q(K)R = 0$ and $\sqrt{t} = R$. In this case there is nothing to prove. Therefore, we may assume $K \subseteq R$. Then the condition $Q(K)\mathfrak{A} \cap Q(K) \cdot 1_R = 0$ is equivalent to $\mathfrak{A} \cap K = 0$.

Let $\mathfrak{U} \cap K = 0$. Then we have a commutative diagram:

$$\begin{array}{ccc} R^{p^n} & \xrightarrow{f} & R \otimes_K Q(K) \\ \downarrow \pi & & \downarrow \pi \otimes 1 \quad (\pi \text{ is the canonical surjection}). \\ R^{p^n}/\mathfrak{U}^{p^n} & \xrightarrow{\bar{f}} & R/\mathfrak{U} \otimes_K Q(K) \end{array}$$

As in Lemma 1, it holds $f(R^{p^n}) \subseteq Q(K)$ and $\pi \otimes 1$ is bijective on $f(R^{p^n})$. Hence we have $\mathfrak{U}^{p^n} \subset \ker \pi \bar{f} = \ker(\pi \otimes 1)f = \ker f = t(n) \subseteq \sqrt{t}$. Therefore, we have $\mathfrak{U} \subseteq \sqrt{t}$ by Lemma 1. Q.E.D.

THEOREM 4. *Let K be a commutative ring of characteristic $p > 0$ and let R be a K -algebra with the structure homomorphism ϕ . Let $\text{Spec}(K/R) = \{\mathfrak{p} \in \text{Spec}(K); \phi^{-1}(\mathfrak{p}R) = \mathfrak{p}\}$ and let ψ be a mapping of $\text{Spec}(R)$ into $\text{Spec}(K)$ such that $\psi(\mathfrak{P}) = \phi^{-1}(\mathfrak{P})$ for $\mathfrak{P} \in \text{Spec}(R)$. If R is a purely inseparable K -algebra of finite exponent, then:*

- (1) ψ is a bijection of $\text{Spec}(R)$ onto $\text{Spec}(K/R)$ and
- (2) $\psi^{-1}(\mathfrak{p}) = \{x \in R; ax^{p^m} \in \mathfrak{p}R \text{ for some element } a \in K - \mathfrak{p} \text{ and for some integer } m\}$.

PROOF. It is trivial that $\text{Im } \psi \subseteq \text{Spec}(K/R)$. For any element \mathfrak{p} of $\text{Spec}(K/R)$, $R/\mathfrak{p}R$ is a purely inseparable K/\mathfrak{p} -algebra of finite exponent. Let t be the K/\mathfrak{p} -torsion submodule of $R/\mathfrak{p}R$ and let π be the natural surjection of R onto $R/\mathfrak{p}R$. Let

$$\mathfrak{P} = \{x \in R; rx^{p^m} \in \mathfrak{p}R \text{ for some element } r \in K - \mathfrak{p} \text{ and for some integer } m\}.$$

Then we have $\mathfrak{P} = \pi^{-1}(\sqrt{t})$. Since $\mathfrak{p} \in \text{Spec}(K/R)$, we may $k/\mathfrak{p} \subseteq R/\mathfrak{p}R$. \mathfrak{P} is a prime ideal by Lemma 1 and $\psi(\mathfrak{P}) = \phi^{-1}(\mathfrak{P}) = \mathfrak{p}$ from $k/\mathfrak{p} \subseteq R/\mathfrak{p}R$. Hence ψ is surjective. Let $\mathfrak{P}' \in \text{Spec}(R)$ and $\psi(\mathfrak{P}') = \mathfrak{p}$. Then, for every element $y \in \mathfrak{P}$, we have $rx^{p^m} \in \mathfrak{P}'$ for some element $r \in K - \mathfrak{p}$ and for some integer m . From $\phi(r) \notin \mathfrak{P}'$, it holds $y \in \mathfrak{P}'$ and hence $\mathfrak{P} \subseteq \mathfrak{P}'$. Let us prove the converse inclusion. Let $x \in \mathfrak{P}'$ and $\pi(x) \in K/\mathfrak{p}$. Then $x \in \pi^{-1}(K/\mathfrak{p}) = \phi(K) + \mathfrak{p}R$ and hence we can write $x = \phi(a) + s$, for $a \in K$, $s \in \mathfrak{p}R \subseteq \mathfrak{P}'$. Therefore, $\phi(a) \in \mathfrak{P}'$ and hence $\pi(x) = 0$. Thus we have $\pi(\mathfrak{P}') \cap K/\mathfrak{p} = 0$. By Lemma 2, $\pi(\mathfrak{P}') \subseteq \sqrt{t}$ and hence we have $\mathfrak{P}' \subseteq \pi^{-1}(\sqrt{t}) = \mathfrak{P}$. Therefore, ψ is injective. This completes the proof. Q.E.D.

LEMMA 3. *Let K be a commutative ring of characteristic $p > 0$ and let R be a purely inseparable K -algebra of finite exponent with the structure homomorphism ϕ . If R is a projective K -module, then we have $\text{Spec}(K) = \text{Spec}(K/R)$.*

PROOF. Let $\mathfrak{p} \in \text{Spec}(K)$. Suppose $\phi^{-1}(\mathfrak{p}R) \neq \mathfrak{p}$. Then there is an element $x \in \phi^{-1}(\mathfrak{p}R) - \mathfrak{p}$. We can write $\phi(x) = \sum a_i x_i$, $a_i \in \mathfrak{p}$, $x_i \in R$. Since $R \otimes K_{\mathfrak{p}}$ is a projective $K_{\mathfrak{p}}$ -module, we have $\mathfrak{p}K_{\mathfrak{p}}(R \otimes K_{\mathfrak{p}}) \neq R \otimes K_{\mathfrak{p}}$ by [1, Proposition 2.7]. Thus we have $x_i^{p^n} \in K_{\mathfrak{p}} \cdot 1_R$ and hence $x^{p^n} \cdot 1_R \in \mathfrak{p}K_{\mathfrak{p}} \cdot 1_R$ by Theorem 3. Therefore, it holds $x^{p^n} \in \mathfrak{p}K_{\mathfrak{p}}$. On the other hand, x is a unit in $K_{\mathfrak{p}}$. This is a contradiction. Q.E.D.

We have the following theorem by Theorem 4 and Lemma 3.

THEOREM 5. *Let K be a commutative ring of characteristic $p > 0$ and let R be a purely inseparable K -algebra of finite exponent with the structure homomorphism ϕ . If R is a projective K -module, then ψ is a bijection of $\text{Spec}(R)$ onto $\text{Spec}(K)$ where $\psi(\mathfrak{P}) = \phi^{-1}(\mathfrak{P})$ for $\mathfrak{P} \in \text{Spec}(R)$.*

THEOREM 6. *Let K be a commutative ring of characteristic $p > 0$ and let R be a purely inseparable K -algebra of finite exponent. If R is a projective K -module, then R is a faithfully flat K -module.*

PROOF. It follows from Theorem 5 and [4, Theorem 4.D]. Q.E.D.

THEOREM 7. *Let K be a commutative ring of characteristic $p > 0$ and let R be a finite flat K -module. If R is a purely inseparable K -algebra with the structure homomorphism ϕ , then (1) $\text{Spec}(K) = \text{Spec}(K/R)$, (2) ψ is a bijection of $\text{Spec}(R)$ onto $\text{Spec}(K)$ where $\psi(\mathfrak{P}) = \phi^{-1}(\mathfrak{P})$ for $\mathfrak{P} \in \text{Spec}(R)$, and (3) R is a faithfully flat K -module.*

PROOF. Since $R \otimes K_p$ is a finite K_p -module, we have $\mathfrak{p}K_p(R \otimes K_p) \neq R \otimes K_p$ by Nakayama's Lemma. Therefore, (1) and (2) are proved the same way as in Lemma 3. Therefore, (3) follows from [4, Theorem 4.D]. Q.E.D.

2. P.H.D. rings. Let K be a commutative ring. Let R be a K -algebra and let M be an R -module. A q th order derivation D of R into M over K is defined as an element of $\text{Hom}_K(R, M)$ such that for any set of $(q + 1)$ -elements x_0, x_1, \dots, x_q of R , we have an identity,

$$D(x_0 x_1 \cdots x_q) = \sum_{s=1}^q (-1)^{s-1} \sum_{i_1 < \cdots < i_s} x_{i_1} \cdots x_{i_s} D(x_0 \cdots \check{x}_{i_1} \cdots \check{x}_{i_s} \cdots x_q).$$

$\mathcal{D}_K(R, M)$ denotes the derivation module of R into M over K [5].

The following theorem is seen in [3].

THEOREM 8. *Let K be a commutative ring. Let R be a K -algebra and let M be an R -module. Then the following are equivalent.*

- (1) $\mathcal{D}_K(R, M) = \text{Hom}_K(R, M)$.
- (2) Each element of $\text{Hom}_K(R, M)$ which vanishes on K is a K -derivation of some order.
- (3) Each element of $\text{Hom}_R(I_{R/K}, M)$ vanishes on some power $I_{R/K}^{n+1}$ of $I_{R/K}$.

DEFINITION 3. Let K be a commutative ring and let R be a K -algebra. R is called a P.H.D. ring over K with respect to an R -module M if $\mathcal{D}_K(R, M) = \text{Hom}_K(R, M)$. R is called a P.H.D. ring over K if R is a P.H.D. ring over K with respect to all R -modules M .

EXAMPLE 2. Let K be a commutative ring and let S be a multiplicatively closed subset of K . Then K_S is a P.H.D. ring over K .

EXAMPLE 3. Let $R = K[X]$ be a polynomial ring over a commutative ring K . Then R is not a P.H.D. ring over K with respect to R .

THEOREM 9. Let K be a commutative ring and let R be a K -algebra. Then the following are equivalent.

- (1) R is a P.H.D. ring over K .
- (2) R is a P.H.D. ring over K with respect to $R \otimes_K R$.
- (3) $I_{R/K}^n = 0$ for some integer n .
- (4) For any K -algebra T and for any K -algebra homomorphisms u_1, u_2 of R into T , there exists an integer n such that $\prod_{i=1}^n (u_1(x_i) - u_2(x_i)) = 0$ for any n -elements $x_1, x_2, \dots, x_n \in R$.

PROOF. (1) \rightarrow (2) and (3) \rightarrow (1) are trivial. (2) \rightarrow (3) follows from $1 \in \text{Hom}_R(I_{R/K}, R \otimes_K R)$ and Theorem 8, [3]. Let us set $T = R \otimes_K R$ and $u_1(x) = x \otimes 1$, $u_2(x) = 1 \otimes x$ in (4). Then (4) implies (3). It remains to be shown that (3) implies (4). Suppose (4) is not satisfied. Then there exist a K -algebra T and K -algebra homomorphisms u_1, u_2 of R into T such that for any integer n it holds that $\prod_{i=1}^n (u_1(x_i) - u_2(x_i)) \neq 0$ for some elements x_i of R . Let us define a mapping $u_1 \otimes u_2$ of $R \otimes_K R$ into T such that $(u_1 \otimes u_2)(x \otimes y) = u_1(x)u_2(y)$. Then $u_1 \otimes u_2$ is a K -algebra homomorphism. Then since $I_{R/K}^n = 0$ we have $(u_1 \otimes u_2)(I_{R/K}^n) = 0$. On the other hand we have

$$0 = (u_1 \otimes u_2)\left(\prod (x_i \otimes 1 - 1 \otimes x_i)\right) = \prod (u_1(x_i) - u_2(x_i)) \neq 0.$$

This is a contradiction. Hence the implication (3) \rightarrow (4) is good. Q.E.D.

REMARK. It is trivial that a P.H.D. ring over K is a purely inseparable K -algebra of finite exponent.

COROLLARY. Let k be a commutative field of characteristic $p > 0$ and let K be a field extension. If K is a P.H.D. ring over k , then K is finite over k .

PROOF. Suppose K is not finite over k . Then, there exists a p -basis $\{x_i\}_{i \in I}$ of K over k such that I is an infinite set. Then we have $\prod_{i=1}^n (x_i \otimes 1 - 1 \otimes x_i) \neq 0$ for any integer n by [2, §2, Proposition 13]. This is a contradiction by Theorem 9. Q.E.D.

THEOREM 10. Let K be a commutative ring of characteristic $p > 0$. Let R be a K -algebra and let S be a multiplicatively closed subset of K such that every element of S is not a zero divisor of R . Then (1) R is a P.H.D. ring over K if and only if $S^{-1}R$ is a P.H.D. ring over $S^{-1}K$. (2) R is a P.H.D. ring over K with respect to R if $S^{-1}R$ is a P.H.D. ring over $S^{-1}K$ with respect to $S^{-1}R$. (3) If R is a finite K -module, then the converse of (2) is true.

PROOF. Since $I_{S^{-1}R/S^{-1}K} = S^{-1}I_{R/K}$, the "only if" part is obvious. Conversely, suppose $I_{S^{-1}R/S^{-1}K}^n = 0$. Let $\{v_i\}$ be a system of generators of R . For any element $\sum t_i \otimes v_i \in (I_{R/K})^n$ we have $\sum t_i \otimes v_i = 0$ in $I_{S^{-1}R/S^{-1}K}^n$, so that there exists an element $s \in S$ such that $s(\sum t_i \otimes v_i) = 0$ in $I_{R/K}^n$. By the lemma of Bourbaki [2, §2, Lemma 10], there exist elements $a_{ij} \in K$ and $\alpha_i \in R$ ($1 \leq i \leq m$, $1 \leq j \leq n$) such that

- (a) $\sum a_{ij} s v_j = 0$ for all i .
- (b) $t_j = \sum a_{ij} \alpha_i$ for all j .

Since s is not a zero divisor of R , (a) implies $\sum a_{ij}v_j = 0$ and hence $\sum t_i \otimes v_i = 0$ in $I_{R/K}$. Therefore, $I_{R/K}^n = 0$.

This completes the proof of (1). Let us prove (2). For each element $f \in \text{Hom}_R(I_{R/K}, R)$, let us define a mapping $\tilde{f} \in \text{Hom}_{S^{-1}R}(S^{-1}I_{R/K}, S^{-1}R)$ such that $\tilde{f}(x/s) = (1/s)f(x)$ for $s \in S$, $x \in I_{R/K}$. Then by assumption we have $\tilde{f}(I_{S^{-1}R/S^{-1}K}^n) = 0$ for some integer n . Therefore, for any element $x \in I_{R/K}^n$ there exist elements $s \in S$ such that $sf(x) = 0$ in R . Since s is not a zero divisor of R , we have $f(x) = 0$ in R . (3) Let $f \in \text{Hom}_{S^{-1}K}(S^{-1}R, S^{-1}R)$ with $f(1) = 0$. Since R is finite over K , there exists an element s of K such that $sf \in \text{Hom}_K(R, R)$. By assumption sf is a high order derivation of R . Therefore, f is a high order derivation of $S^{-1}R$. Q.E.D.

PROPOSITION 5. *Let K be a commutative ring of characteristic $p > 0$. (1) Let R, R' be K -algebras where $\phi: R \rightarrow R'$ is a surjective map of K -algebras. Then, if R is a P.H.D. ring over K , R' is a P.H.D. ring over K . (2) Let R, S be K -algebras and $S \rightarrow R$ be a K -algebra homomorphism. Then, if R is a P.H.D. ring over K , R is a P.H.D. ring over S .*

PROOF. (1) We consider the following commutative exact diagram.

$$\begin{array}{ccccccccc} 0 & \rightarrow & I_{R/K} & \rightarrow & R \otimes_K R & \rightarrow & R & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & I_{R'/K} & \rightarrow & R' \otimes_K R' & \rightarrow & R' & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Then, if $I_{R/K}$ is nilpotent, $I_{R'/K}$ is nilpotent. (2) is trivial from Theorem 9. Q.E.D.

THEOREM 11. *Let K be a commutative ring of characteristic $p > 0$. Let S be a K -algebra and let R be an S -algebra. Then, if S is a P.H.D. ring over K and R is a P.H.D. ring over S , R is a P.H.D. ring over K .*

PROOF. We consider the following commutative exact diagram.

$$\begin{array}{ccccccccc} 0 & \rightarrow & I_{S/K} & \rightarrow & S \otimes_K S & \rightarrow & S & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & I_{R/K} & \rightarrow & R \otimes_K R & \rightarrow & R & \rightarrow & 0 \\ & & \downarrow & & \downarrow v & & \downarrow & & \\ 0 & \rightarrow & I_{R/S} & \rightarrow & R \otimes_S R & \rightarrow & R & \rightarrow & 0 \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array} \quad (v \text{ is the canonical map}).$$

By assumption we have $I_{S/K}^n = 0$, $I_{R/S}^n = 0$ for some integer n . Since v is a ring homomorphism, we have $v(I_{R/K}^n) = v(I_{R/K})^n = I_{R/S}^n = 0$. Thus $I_{R/K}^n \subseteq \ker v = I_{S/K}(R \otimes_K R)$ and hence $I_{R/K}$ is nilpotent. Thus R is a P.H.D. ring over K . Q.E.D.

PROPOSITION 6. *Let K be a commutative ring of characteristic $p > 0$. Let R be a K -algebra and let \mathfrak{A} be an ideal of R . Suppose R is a projective K -module. Then, if R is a P.H.D. ring over K with respect to R , R/\mathfrak{A} is a P.H.D. ring over $K/(\mathfrak{A} \cap K)$ with respect to R/\mathfrak{A} .*

PROOF. Let π be the canonical mapping of R onto R/\mathfrak{A} and let f be an element of $\text{Hom}_{K/(\mathfrak{A} \cap K)}(R/\mathfrak{A}, R/\mathfrak{A})$ such that $f(1_R) = \bar{0}$ in R/\mathfrak{A} . Since R is K -projective, there exists $g \in \text{Hom}_K(R, R)$ such that $\pi g = f\pi$. Let $g(1_R) = \alpha$. Then $\pi(\alpha) = 0$ and hence $\alpha \in \mathfrak{A}$. By assumption $g - \alpha \cdot 1_R$ is an n th order derivation of R for some integer n . For any set of $(n+1)$ -elements $\bar{x}_0, \bar{x}_1, \dots, \bar{x}_n$ ($x_i \in R$) of R/\mathfrak{A} , we have

$$\begin{aligned} & \pi(g - \alpha \cdot 1_R)(x_0 x_1 \cdots x_n) \\ &= \sum_{s=1}^n (-1)^{s-1} \sum_{i_1 < \cdots < i_s} \bar{x}_{i_1} \cdots \bar{x}_{i_s} \pi(g - \alpha \cdot 1_R)(x_0 \cdots \check{x}_{i_1} \cdots \check{x}_{i_s} \cdots x_n). \end{aligned}$$

Therefore, it holds

$$\begin{aligned} & f(\bar{x}_0 \bar{x}_1 \cdots \bar{x}_n) - \bar{\alpha}(\bar{x}_0 \bar{x}_1 \cdots \bar{x}_n) \\ &= \sum_{s=1}^n (-1)^{s-1} \sum_{i_1 < \cdots < i_s} \bar{x}_{i_1} \cdots \bar{x}_{i_s} f(\bar{x}_0 \cdots \check{x}_{i_1} \cdots \check{x}_{i_s} \cdots \bar{x}_n) \\ &\quad - \sum_{s=1}^n (-1)^{s-1} \sum_{i_1 < \cdots < i_s} \bar{x}_{i_1} \cdots \bar{x}_{i_s} \bar{\alpha}(\bar{x}_0 \cdots \check{x}_{i_1} \cdots \check{x}_{i_s} \cdots \bar{x}_n). \end{aligned}$$

Since $\bar{\alpha} = 0$, f is an n th order derivation of R/\mathfrak{A} . Q.E.D.

It is trivial that if R is a P.H.D. ring over K , then R is a P.H.D. ring over K with respect to R . The converse is true in the following case.

THEOREM 12. *Let K be a commutative ring of characteristic $p > 0$ and let R be a K -algebra. Suppose R is a free K -module. Then, if R is a P.H.D. ring over K with respect to R , R is a P.H.D. ring over K .*

PROOF. Let $\{y_j\}_{j \in J}$ be a free base of R over K . Suppose R is not a P.H.D. ring over K . Then we have $I_{R/K}^n \neq 0$ for any integer n and hence there exists a nonzero element z_n in $I_{R/K}^n$ for every n . Let us fix $\{z_n\}$. Since $\{y_j\}_{j \in J}$ is a free basis of R , every z_n is represented by $\{y_j\}_{j \in J}$. The set of $\{y_j\}_{j \in J}$ denoting $\{z_n\}$ is countable and is totally ordered. Let us denote this set $\{x_i\}$. Then it holds that

$$I_{R/K} = \sum R(x_i \otimes 1 - 1 \otimes x_i) \oplus S \quad (\text{direct sum as } R\text{-modules}).$$

Let $|x_i|$ denote the maximum of n such that x_i is contained in the representation of z_n . (1) The case in which every $|x_i|$ is finite. Let

$$z_1 = \sum a_{n(1)}(x_{n(1)} \otimes 1 - 1 \otimes x_{n(1)}).$$

Then the set of z_n containing at least one of $\{x_{n(1)} \otimes 1 - 1 \otimes x_{n(1)}\}_{n(1)}$ is finite. Thus there is an integer N_1 such that, for all $n > N_1$, z_n does not contain any of $\{x_{n(1)} \otimes 1 - 1 \otimes x_{n(1)}\}_{n(1)}$. Let $O(1) = \min n(1)$. Next let

$$z_{N_1} = \sum a_{n(N_1)}(x_{n(N_1)} \otimes 1 - 1 \otimes x_{n(N_1)}).$$

Since the set of z_n containing at least one of $\{x_{n(N_1)} \otimes 1 - 1 \otimes x_{n(N_1)}\}_{n(N_1)}$ is finite, there is an integer N_2 such that, for all $n \geq N_2$, z_n does not contain any of $\{x_{n(N_1)} \otimes 1 - 1 \otimes x_{n(N_1)}\}_{n(N_1)}$. Let $O(2) = \min n(N_1)$. Continuing this process, we can obtain a sequence, $O(1), O(2), \dots$. Let us define a homomorphism of $I_{R/K}$ into R such that

$$\begin{cases} f(x_{O(i)} \otimes 1 - 1 \otimes x_{O(i)}) = 1 & \text{for all } i, \\ f(x_n \otimes 1 - 1 \otimes x_n) = 0 & \text{for } n \neq O(i), \\ f(S) = 0. \end{cases}$$

Then $f(I_{R/K}^n) \neq 0$ for all integers n and, hence, R is not a P.H.D. ring over K with respect to R . This is a contradiction. (2) The case in which at least one of $|x_i|$ is infinite. Let us set $|x_1| = \infty$. Then we define a homomorphism of $I_{R/K}$ into R such that

$$\begin{cases} f(x_1 \otimes 1 - 1 \otimes x_1) = 1, \\ f(x_i \otimes 1 - 1 \otimes x_i) = 0 & \text{for } i \neq 1, \\ f(S) = 0. \end{cases}$$

Then $f(I_{R/K}^n) \neq 0$ for all integers n and this is a contradiction. Therefore, R is a P.H.D. ring over K . Q.E.D.

THEOREM 13. *Let K be a commutative ring of characteristic $p > 0$ and let R be a K -algebra. Suppose R is a projective K -module. If R is a P.H.D. ring over K with respect to R , then R is a purely inseparable K -algebra.*

PROOF. Let T be a reduced K -algebra and let u_1, u_2 be K -algebra homomorphisms of R into T . Since $\text{Im}(u_1 - u_2) \stackrel{\tau}{\cong} R/R_0$ for some K -submodule R_0 of R , we obtain a diagram.

$$\begin{array}{ccccc} & & R & & \\ & & \downarrow \tau(u_1 - u_2) & & (\pi: \text{the canonical mapping}). \\ R & \xrightarrow{\pi} & R/R_0 & \rightarrow & 0 \end{array}$$

Since R is a K -projective, there exists a homomorphism g of R into R such that $\pi g = \tau(u_1 - u_2)$. Let $g(1) = \alpha$. Then $\alpha \in R_0$. By assumption $g - \alpha 1$ is an n th order derivation for some integer n . For an integer t with $p^t > n$ we have $(g - \alpha 1)(x^{p^t}) = 0$ for any element $x \in R$. Thus $\pi g(x^{p^t}) = 0$ and hence $\tau(u_1 - u_2)(x^{p^t}) = 0$. Since τ is bijective, we have $(u_1 - u_2)(x^{p^t}) = 0$ and hence $(u_1(x) - u_2(x))^{p^t} = 0$. Since T is reduced, we have $u_1 = u_2$. Therefore, R is a purely inseparable K -algebra. Q.E.D.

The assumption " R is a projective K -module" in Proposition 6 and Theorem 13 is essential. Let us give an example.

EXAMPLE 4. Let Z be the ring of integers and let X, Y and W be indeterminates. Let us set $K = (Z/2Z)[X, Y]_{(X, Y)}$ and $R = K[W]/(XW - Y)$. Then (1) K is a local domain with the maximal ideal $\mathfrak{M} = (X, Y)K$ and R is a K -algebra injectively containing K , (2) R is not K -projective, (3) R is not a purely inseparable K -algebra and (4) R is a P.H.D. ring over K with respect to R . Because, for $\alpha \in K$

a relation $\alpha \cdot 1 = 0$ in R implies $\alpha = (\sum_{i=0}^N a_i W^i)(XW - Y)$ for $a_i \in K$, $a_N \neq 0$. From this equation, W is algebraic over K and this is a contradiction. Therefore we have (1). Let us prove (2). Suppose R is K -projective. Since K is a local ring, R is a free K -module. Let $\{f_i\}$ be a free base of R and let \bar{W} be the residue class of W . We can write

$$1 = \sum \frac{a_i}{1 + \alpha} f_i, \quad \bar{W} = \sum \frac{b_i}{1 + \beta} f_i$$

for $a_i, b_i, \alpha, \beta \in (Z/2Z)[X, Y]$, $\alpha, \beta \in (X, Y)(Z/2Z)[X, Y]$. From the first formula, in the set $\{a_i\}$ there is an element a_{i_0} such that $a_{i_0} = 1 + t$ for $t \in (X, Y)(Z/2Z)[X, Y]$. Then from $X\bar{W} - Y = 0$ it holds that

$$X \sum \frac{b_i}{1 + \beta} f_i - Y \sum \frac{a_i}{1 + \alpha} f_i = 0.$$

Since $\{f_i\}$ is free, we have $Xb_i/(1 + \beta) = Ya_i/(1 + \alpha)$ for all i and especially $Xb_{i_0}/(1 + \beta) = Ya_{i_0}/(1 + \alpha)$. Therefore, we have $X(1 + \alpha)b_{i_0} = (1 + \beta)(1 + t)Y$ and so $Y = X(1 + \alpha)b_{i_0} - (\beta + t + \beta t)Y$. This shows X, Y are algebraic over $Z/2Z$. This is a contradiction and (2) is proved. Next, suppose R is a purely inseparable K -algebra. Then there is an integer n satisfying $\bar{W}^{2^n} \otimes 1 - 1 \otimes \bar{W}^{2^n} = 0$. By the lemma of Bourbaki [2, §2, Lemma 10], there are elements $a_{ij} \in K$ and $X_j \in R$ such that $\sum a_{ij} \bar{W}^i = 0$ for all j , and $1 = \sum a_{2^n j} X_j$. From the second formula, in the set $\{a_{2^n j}\}$ there exists at least one element having the form $1 + \alpha$, $\alpha \in \mathfrak{M}$ (say $a_{2^n j_0}$). Thus from the first formula, we have

$$\bar{W} = - \sum_{i \neq 2^n} a_{ij_0} \bar{W}^i - \alpha \bar{W}^{2^n}$$

and hence

$$W^{2^n} = - \sum_{i \neq 2^n} a_{ij_0} W^i + \sum b_i (XW - Y)^i + \alpha W^{2^n}.$$

This shows that W is algebraic over K and this is a contradiction, which proves (3). At last, we shall show $\text{Hom}_R(I_{R/K}, R) = 0$ and which shows (4). Let f be an element of $\text{Hom}_R(I_{R/K}, R)$ and let us set

$$f(\bar{W}^i \otimes 1 - 1 \otimes \bar{W}^i) = \frac{1}{1 + \alpha} \left[\sum_{i=1}^N (1 + \alpha_i) g_i(Y) \bar{W}^i + c \right],$$

for $c, \alpha, \alpha_i \in (Z/2Z)[X, Y]$, $g_i \in (Z/2Z)[Y]$ and $\alpha, \alpha_i \in (X, Y)(Z/2Z)[X, Y]$. Suppose $g_N(Y) \neq 0$. Since $X^i(\bar{W}^i \otimes 1 - 1 \otimes \bar{W}^i) = 0$, we have

$$0 = X^i \frac{1}{1 + \alpha} \left[\sum (1 + \alpha_i) g_i(Y) \bar{W}^i + c \right].$$

Thus it holds that $(1 + \alpha_i) g_i(Y) \bar{W}^i + c = 0$ and hence we can write

$$(1 + \alpha_i) g_i(Y) \bar{W}^i + c = \frac{h(W)}{1 + \beta} (XW - Y)$$

for some $h(W) \in (Z/2Z)[X, Y][W]$, $\beta \in (X, Y)(Z/2Z)[X, Y]$. This shows W is algebraic over $(Z/2Z)[X, Y]$ and this is a contradiction. Therefore, $g_N(Y) = 0$ and

hence $f(\overline{W}^i \otimes 1 - 1 \otimes \overline{W}^i) = c/(1 + \alpha)$. Since $f[X^i(\overline{W}^i \otimes 1 - 1 \otimes \overline{W}^i)] = 0$, we have $c = 0$ and $f = 0$. This shows $\text{Hom}_R(I_{R/K}, R) = 0$. This completes the proof. Q.E.D.

COROLLARY (Y. NAKAI [6]). *Let K be a commutative ring of characteristic $p > 0$ and let R be a K -algebra. Suppose K is a field. If R is a P.H.D. ring over K with respect to R , then we have the following.*

- (1) R is a purely inseparable K -algebra.
- (2) R is a quasi-local ring.
- (3) The maximal ideal \mathfrak{M} of R is nilpotent.
- (4) The residue field R/\mathfrak{M} is a finite extension of K .

PROOF. (1) follows from Theorem 13 and thus (2) follows from Proposition 2, Corollary. By Theorem 12, R is a P.H.D. ring over K and hence R/\mathfrak{M} is a P.H.D. ring over K by Proposition 5. Therefore, (4) follows from Theorem 9, Corollary. Since $I_{R/K}^n = 0$, we have $x_1 x_2 \cdots x_{p^n} = 0$ (for any $x_1, x_2, \dots, x_{p^n} \in \mathfrak{M}$) by [2, §2, Proposition 13]. Therefore, (3) is proved. Q.E.D.

Let K be a commutative ring and let R be a K -algebra. A higher K -derivation $\Delta = \{1, D_1, D_2, \dots, D_m\}$ of rank m is a set of linear K -endomorphisms of R into itself satisfying the conditions

$$D_n(xy) = \sum_{i=0}^n D_i(x) D_{n-i}(y) \quad (n \leq m) \text{ where } D_0 = 1.$$

PROPOSITION 7. *Let K be a commutative ring of characteristic $p > 0$ and let R be a K -algebra. Let $\Delta = \{D_i\}$ be a higher K -derivation of infinite rank of R . If R is a P.H.D. ring over K , then $\text{Im } D_i$ is nil for every i .*

PROOF. Let us set $\Phi = \sum T^i D_i$ where $T^0 = 1$, $D_0 = 1$. Then Φ is a ring homomorphism of R into $R[[T]]$. Since $(\Phi - 1)(1) = 0$, it follows that $\Phi - 1$ is a high order (for example n th order) derivation of R into $R[[T]]$ by assumption. For $p' > n$, we have $(\Phi - 1)(x^{p'}) = 0$ for any element x of R . Thus it holds that

$$0 = (\Phi - 1)(x^{p'}) = D_1(x)^{p'} T^{p'} + D_2(x)^{p'} T^{2p'} + \dots$$

Therefore, we have $D_i(x)^{p'} = 0$ for all i . Q.E.D.

We shall give a structure theorem for a finite flat P.H.D. ring R over K with respect to R .

THEOREM 14. *Let K be a commutative ring of characteristic $p > 0$ and let R be a finite flat K -module. If R is a P.H.D. ring over K with respect to R with the structure homomorphism ϕ , then (1) ψ is bijective, (2) $\psi^{-1}(\mathfrak{p}) = \{x \in R; sx^{p^n} \in \mathfrak{p}R \text{ for some } s \in K - \mathfrak{p} \text{ and for some integer } n\}$ where $\psi(\mathfrak{P}) = \phi^{-1}(\mathfrak{P})$ for $\mathfrak{P} \in \text{Spec}(R)$, and (3) R is a faithfully flat K -module.*

PROOF. Let $\mathfrak{p} \in \text{Spec}(K)$. Then $R \otimes K_{\mathfrak{p}}$ is a finite flat $K_{\mathfrak{p}}$ -module and hence $R \otimes K_{\mathfrak{p}}$ is a free $K_{\mathfrak{p}}$ -module by a well-known theorem. Thus $R \otimes K_{\mathfrak{p}}$ is a P.H.D. ring over $K_{\mathfrak{p}}$ by Theorem 12. Therefore, by Theorem 10 R is a P.H.D. ring over K and hence we complete the proof by Theorem 7. Q.E.D.

REFERENCES

1. H. Bass, *Finitistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc. **3** (1960), 466–488.
2. N. Bourbaki, *Eléments de mathématique, Algèbre commutative*, Chapitres 1 et 2, Hermann, Paris, 1961.
3. T. Kikuchi, *Some remarks on high order derivations*, J. Math. Kyoto Univ. **11** (1971), 71–87.
4. M. Matsumura, *Commutative algebra*, Benjamin, New York, 1970.
5. Y. Nakai, *High order derivations*. I, Osaka J. Math. **7** (1970), 1–27.
6. ———, *On a ring with plenty of high order derivations*, J. Math. Kyoto Univ. **13** (1972), 159–164.
7. P. Samuel, *Séminaire d'algèbre commutative: 1967/1968. Les épimorphismes d'anneaux*, Secrétariat Math., Paris, 1968.
8. M. E. Sweedler, *Purely inseparable algebras*, J. Algebra **35** (1975), 342–355.

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